GradedRing-ForHomalg

Endow Commutative Rings with an Abelian Grading

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Chapter 1

Introduction

1.1 What is the Role of the GradedRingForHomalg Package in the homalg Project?

The homalg project [hpa10] aims at providing a general and abstract framework for homological computations. The package GradedRingForHomalg enables the homalg project to endow commutative rings in homalg with an Abelian grading.

1.2 Functionality

The package GradedRingForHomalg on the one hand builds on the package MatricesForHomalg and on the other hands adds functionality to MatricesForHomalg.

Chapter 2

Installation of the GradedRingForHomalg Package

To install this package just extract the package's archive file to the GAP `pkg` directory. GradedRingForHomalg also needs the package homalg.

By default the GradedRingForHomalg package is not automatically loaded by GAP when it is installed. You must load the package with

```
LoadPackage("GradedRingForHomalg");
```

before its functions become available.

Please, send me us e-mail if you have any questions, remarks, suggestions, etc. concerning this package. Also, we would be pleased to hear about applications of this package.

Mohamed Barakat and Markus Lange-Hegermann

Chapter 3

Graded Rings

The package GradedRingForHomalg defines the classes of graded rings, ring elements and matrices over such rings. These three objects can be used as data structures defined in MatricesForHomalg on which the homalg project can rely to do homological computations over graded rings.

The graded rings most prominently can be used with methods known from general homalg rings. The methods for doing the computations are presented in the appendix (B), since they are not for external use. The new attributes and operations are documented here.

Since the objects inplemented here are representations from objects elsewhere in the homalg project (i.e. MatricesForHomalg), we want to stress that there are many other operations in MatricesForHomalg, which can be used in connection with the ones presented here. A few of them can be found in the examples and the appendix of this documentation.

Operations within MatricesForHomalg that take matrices as input and produce a matrix as an output produce homogeneous output for homogeneous input in the following cases: the graded ring in question is either a polynomial ring or the exterior algebra residing in Singular, and the called operation is one of the following listed below:

- `SyzygiesGeneratorsOfRows`
- `SyzygiesGeneratorsOfColumns`
- `ReducedSyzygiesGeneratorsOfRows`
- `ReducedSyzygiesGeneratorsOfColumns`
- `BasisOfRowModule`
- `BasisOfColumnModule`
- `ReducedBasisOfRowModule`
- `ReducedBasisOfColumnModule`
- `DecideZeroRows`
- `DecideZeroColumns`
- `LeftDivide`
- `RightDivide`

These operation trigger Gröbner bases computations in Singular, which are always forced to be performed with a tail reduction by homalg. In particular, the resulting elements of the Gröbner bases have to be homogeneous.

3.1 Graded Rings: Category and Representations

3.1.1 IsHomalgGradedRingRep

▷ `IsHomalgGradedRingRep(R)` (Representation)

Returns: true or false

The representation of homalg graded rings.

(It is a subrepresentation of the GAP representation `IsHomalgRingOrFinitelyPresentedModuleRep`.)

Code
```
DeclareRepresentation( "IsHomalgGradedRingRep",
        IsHomalgGradedRing and
        IsHomalgGradedRingOrGradedModuleRep,
        [ "ring" ] );
```

3.1.2 IsHomalgGradedRingElementRep

▷ `IsHomalgGradedRingElementRep(r)` (Representation)

Returns: true or false

The representation of elements of homalg graded rings.

(It is a representation of the GAP category `IsHomalgRingElement`.)

Code
```
DeclareRepresentation( "IsHomalgGradedRingElementRep",
        IsHomalgGradedRingElement,
        [ ] );
```

3.2 Graded Rings: Constructors

3.2.1 HomalgGradedRingElement (constructor for graded ring elements using numerator and denominator)

▷ `HomalgGradedRingElement(numer, denom, R)` (function)

▷ `HomalgGradedRingElement(numer, R)` (function)

Returns: a graded ring element

Creates the graded ring element *numer*/*denom* or in the second case *numer*/1 for the graded ring *R*. Both *numer* and *denom* may either be a string describing a valid global ring element or from the global ring or computation ring.

3.3 Graded Rings: Attributes and Properties

3.3.1 DegreeGroup

▷ `DegreeGroup(S)` (attribute)

Returns: a left $\mathbb{Z}$-module

The degree Abelian group of the commutative graded ring *S*.

3.3.2 CommonNonTrivialWeightOfIndeterminates

▷ `CommonNonTrivialWeightOfIndeterminates(S)` (attribute)

Returns: a degree

The common nontrivial weight of the indeterminates of the graded ring *S* if it exists. Otherwise an error is issued. WARNING: Since the DegreeGroup and WeightsOfIndeterminates are in some cases bound together, you MUST not set the DegreeGroup by hand and let the algorithm create the weights. Set both by hand, set only weights or use the method WeightsOfIndeterminates to set both. Never set the DegreeGroup without the WeightsOfIndeterminates, because it simply wont work!

3.3.3 WeightsOfIndeterminates

▷ `WeightsOfIndeterminates(S)` (attribute)

Returns: a list or listlist of integers

The list of degrees of the indeterminates of the graded ring *S*.

3.3.4 IsHomogeneousRingElement (for homalg graded ring elements)

▷ `IsHomogeneousRingElement(r)` (operation)

Returns: `true` or `false`

returns whether the graded ring element *r* is homogeneous or not.

3.4 Graded Rings: Operations and Functions

3.4.1 UnderlyingNonGradedRing (for homalg graded rings)

▷ `UnderlyingNonGradedRing(R)` (operation)

Returns: a homalg ring

Internally there is a ring, in which computations take place.

3.4.2 UnderlyingNonGradedRing (for homalg graded ring elements)

▷ `UnderlyingNonGradedRing(r)` (operation)

Returns: a homalg ring

Internally there is a ring, in which computations take place.

3.4.3 Name (for homalg graded ring elements)

▷ `Name(r)` (operation)

Returns: a string

The name of the graded ring element *r*.

3.4.4 HomogeneousPartOfRingElement (for homalg graded ring elements and elements in degree groups)

▷ `HomogeneousPartOfRingElement(r, degree)` (operation)

Returns: a graded ring element

returns the summand of *r* whose monomials have the given degree *degree* and if *r* has no such monomials then it returns the zero element of the ring.

Chapter 4

Homogeneous Matrices

The package GradedRingForHomalg defines the classes of graded rings, ring elements and homogeneous matrices over such rings. These three objects can be used as data structures defined in MatricesForHomalg on which the homalg project can rely to do homological computations over graded rings.

The graded rings most prominently can be used with methods known from general homalg rings. The methods for doing the computations are presented in the appendix (B), since they are not for external use. The new attributes and operations are documented here.

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Operations within MatricesForHomalg that take matrices as input and produce a matrix as an output produce homogeneous output for homogeneous input in the following cases: the graded ring in question is either a polynomial ring or the exterior algebra residing in Singular, and the called operation is one of the following listed below:

- `SyzygiesGeneratorsOfRows`
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- `RightDivide`

These operation trigger Gröbner bases computations in Singular, which are always forced to be performed with a tail reduction by homalg. In particular, the resulting elements of the Gröbner bases have to be homogeneous.

4.1 Homogeneous Matrices: Category and Representations

4.1.1 IsHomalgMatrixOverGradedRingRep

▷ `IsHomalgMatrixOverGradedRingRep(A)` (Representation)

Returns: true or false

The representation of homalg matrices with entries in a homalg graded ring.

(It is a representation of the GAP category `IsMatrixOverGradedRing`.)

Code
```
DeclareRepresentation( "IsHomalgMatrixOverGradedRingRep",
        IsMatrixOverGradedRing,
        [ ] );
```

4.2 Homogeneous Matrices: Constructors

4.2.1 MatrixOverGradedRing (constructor for matrices over graded rings)

▷ `MatrixOverGradedRing(mat, S)` (function)

Returns: a matrix over a graded ring

Creates a matrix for the graded ring *S*, where *mat* is a matrix over `UnderlyingNonGradedRing(S)`.

4.3 Homogeneous Matrices: Attributes

4.3.1 DegreesOfEntries

▷ `DegreesOfEntries(A)` (attribute)

Returns: a listlist of degrees/multi-degrees

The matrix of degrees of the matrix *A*.

4.3.2 NonTrivialDegreePerRow

▷ `NonTrivialDegreePerRow(A[, col_degrees])` (attribute)

Returns: a list of degrees/multi-degrees

The list of first nontrivial degree per row of the matrix *A*.

4.3.3 NonTrivialDegreePerColumn

▷ `NonTrivialDegreePerColumn(A[, row_degrees])` (attribute)

Returns: a list of degrees/multi-degrees

The list of first nontrivial degree per column of the matrix *A*.

4.3.4 HomogeneousPartOfMatrix (for matrices over graded rings and listlist of degrees)

▷ `HomogeneousPartOfMatrix(A, degrees)` (property)

Returns: a homalg matrix over graded ring

The output is the homogeneous part of the matrix *A* with respect to the given degrees *degrees*. See *HomogeneousPartOfRingElement*.

4.3.5 IsMatrixOverGradedRingWithHomogeneousEntries (for matrices over graded rings)

▷ `IsMatrixOverGradedRingWithHomogeneousEntries(A)` (property)

Returns: `true` or `false`

Checks if every entry in a given matrix *A* over a graded ring is homogeneous.

4.4 Homogeneous Matrices: Operations and Functions

4.4.1 UnderlyingNonGradedRing (for matrices over graded rings)

▷ `UnderlyingNonGradedRing(mat)` (operation)

Returns: a homalg ring

The nongraded ring underlying `HomalgRing(mat)`.

4.4.2 SetMatElm (for matrices over graded rings)

▷ `SetMatElm(mat, i, j, r, R)` (operation)

Changes the entry (*i*,*j*) of the matrix *mat* to the value *r*. Here *R* is the graded homalg ring involved in these computations.

4.4.3 AddToMatElm (for matrices over graded rings)

▷ `AddToMatElm(mat, i, j, r, R)` (operation)

Changes the entry (*i*,*j*) of the matrix *mat* by adding the value *r* to it. Here *R* is the (graded) homalg ring involved in these computations.

4.4.4 MatElmAsString (for matrices over graded rings)

▷ `MatElmAsString(mat, i, j, R)` (operation)

Returns: a string

Returns the entry (*i*,*j*) of the matrix *mat* as a string. Here *R* is the (graded) homalg ring involved in these computations.

4.4.5 MatElm (for matrices over graded rings)

▷ `MatElm(mat, i, j, R)` (operation)

Returns: a graded ring element

Returns the entry (*i*,*j*) of the matrix *mat*. Here *R* is the (graded) homalg ring involved in these computations.

Appendix A

The Matrix Tool Operations

The functions listed below are components of the `homalgTable` object stored in the ring. They are only indirectly accessible through standard methods that invoke them.

A.1 The Tool Operations *without* a Fallback Method

There are matrix methods for which homalg needs a `homalgTable` entry for non-internal rings, as it cannot provide a suitable fallback. Below is the list of these `homalgTable` entries.

A.2 The Tool Operations with a Fallback Method

These are the methods for which it is recommended for performance reasons to have a `homalgTable` entry for non-internal rings. homalg only provides a generic fallback method.

A.2.1 MonomialMatrix

▷ `MonomialMatrix(d, R)` (operation)

Returns: a homalg matrix

The column matrix of d-th monomials of the homalg graded ring R.

Example

```
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> S := GradedRing( R );;
gap> m := MonomialMatrix( 2, S );
<A ? x 1 matrix over a graded ring>
gap> NumberRows( m );
6
gap> m;
<A 6 x 1 matrix over a graded ring>
gap> Display( m );
x^2,
x*y,
x*z,
y^2,
y*z,
z^2
(over a graded ring)
```

A.2.2 RandomMatrixBetweenGradedFreeLeftModules

▷ `RandomMatrixBetweenGradedFreeLeftModules(degreesS, degreesT, R)` (operation)

Returns: a homalg matrix

A random $r \times c$-matrix between the graded free *left* modules $R^{(-degreesS)} \to R^{(-degreesT)}$, where r =`Length(degreesS)` and c =`Length(degreesT)`.

Example
```
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "a,b,c";;
gap> S := GradedRing( R );;
gap> rand := RandomMatrixBetweenGradedFreeLeftModules( [ 2, 3, 4 ], [ 1, 2 ], S );
<A 3 x 2 matrix over a graded ring>
gap> #Display( rand );
gap> #a-2*b+2*c,                                        2,
gap> #a^2-a*b+b^2-2*b*c+5*c^2,                          3*c,
gap> #2*a^3-3*a^2*b+2*a*b^2+3*a^2*c+a*b*c-2*b^2*c-3*b*c^2-2*c^3,a^2-4*a*b-3*a*c-c^2
```

A.2.3 RandomMatrixBetweenGradedFreeRightModules

▷ `RandomMatrixBetweenGradedFreeRightModules(degreesT, degreesS, R)` (operation)

Returns: a homalg matrix

A random $r \times c$-matrix between the graded free *right* modules $R^{(-degreesS)} \to R^{(-degreesT)}$, where r =`Length(degreesT)` and c =`Length(degreesS)`.

Example
```
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "a,b,c";;
gap> S := GradedRing( R );;
gap> rand := RandomMatrixBetweenGradedFreeRightModules( [ 1, 2 ], [ 2, 3, 4 ], S );
<A 2 x 3 matrix over a graded ring>
gap> #Display( rand );
gap> #a-2*b-c,a*b+b^2-b*c,2*a^3-a*b^2-4*b^3+4*a^2*c-3*a*b*c-b^2*c+a*c^2+5*b*c^2-2*c^3,
gap> #-5,     -2*a+c,     -2*a^2-a*b-2*b^2-3*a*c
```

A.2.4 Diff

▷ `Diff(D, N)` (operation)

Returns: a homalg matrix

If D is a $f \times p$-matrix and N is a $g \times q$-matrix then $H = Diff(D,N)$ is an $fg \times pq$-matrix whose entry $H[g*(i-1)+j, q*(k-1)+l]$ is the result of differentiating $N[j,l]$ by the differential operator corresponding to $D[i,k]$. (Here we follow the Macaulay2 convention.)

Example
```
gap> S := HomalgFieldOfRationalsInDefaultCAS( ) * "a,b,c" * "x,y,z";;
gap> D := HomalgMatrix( "[ \
> x,2*y,   \
> y,a-b^2, \
> z,y-b    \
> ]", 3, 2, S );
<A 3 x 2 matrix over an external ring>
gap> N := HomalgMatrix( "[ \
> x^2-a*y^3,x^3-z^2*y,x*y-b,x*z-c, \
> x,        x*y,      a-b,  x*a*b  \
> ]", 2, 4, S );
```

```
<A 2 x 4 matrix over an external ring>
gap> H := Diff( D, N );
<A 6 x 8 matrix over an external ring>
gap> Display( H );
2*x,     3*x^2, y,z,  -6*a*y^2,-2*z^2,2*x,0,
1,       y,     0,a*b,0,       2*x,   0,  0,
-3*a*y^2,-z^2,  x,0,  -y^3,    0,     0,  0,
0,       x,     0,0,  0,       0,     1,  b*x,
0,       -2*y*z,0,x,  -3*a*y^2,-z^2,  x+1,0,
0,       0,     0,0,  0,       x,     1,  -a*x
```

Appendix B

Overview of the GradedRingForHomalg Package Source Code

This appendix is included in the documentation to shine some light on the mathematical backgrounds of this Package. Neither is it needed to work with this package nor should the methods presented here be called directly. The functions documented here are entries of the so called ring table and not to be called directly. There are higher level methods in declared and installed in MatricesForHomalg, which call this functions (→ `?MatricesForHomalg:The Basic Matrix Operations`).

B.1 The generic Methods

We will present some methods as an example, to show the idea:

B.1.1 BasisOfRowModule (for graded rings)

▷ `BasisOfRowModule(M)` (function)

Returns: a distinguished basis (i.e. a distinguished generating set) of the module generated by M

Code
```
BasisOfRowModule :=
  function( M )
     return MatrixOverGradedRing(
                    BasisOfRowModule( UnderlyingMatrixOverNonGradedRing( M ) ),
                    HomalgRing( M ) );
  end,
```

B.1.2 DecideZeroRows (for graded rings)

▷ `DecideZeroRows(A, B)` (function)

Returns: a reduced form of *A* with respect to *B*

Code
```
DecideZeroRows :=
  function( A, B )
    return MatrixOverGradedRing(
```

```
                    DecideZeroRows( UnderlyingMatrixOverNonGradedRing( A ),
                            UnderlyingMatrixOverNonGradedRing( B ) ),
                    HomalgRing( A ) );
  end,
```

B.1.3 SyzygiesGeneratorsOfRows (for graded rings)

▷ `SyzygiesGeneratorsOfRows(M)` (function)

Returns: a distinguished basis of the syzygies of the argument

```
SyzygiesGeneratorsOfRows :=
  function( M )
    return MatrixOverGradedRing(
                    SyzygiesGeneratorsOfRows( UnderlyingMatrixOverNonGradedRing( M ) ),
                    HomalgRing( M ) );
  end,
```

B.2 Tools

The package GradedRingForHomalg also implements tool functions. These are referred to from MatricesForHomalg automatically. We list the implemented methods here are and refer to the MatricesForHomalg documentation (→ `?MatricesForHomalg: The Matrix Tool Operations` and `?MatricesForHomalg:RingElement`) for details. All tools functions from MatricesForHomalg not listed here are also supported by fallback tools.

- IsZero
- IsOne
- Minus
- DivideByUnit
- IsUnit
- Sum
- Product
- ShallowCopy
- ZeroMatrix
- IdentityMatrix
- AreEqualMatrices
- Involution
- TransposedMatrix
- CertainRows

- CertainColumns
- UnionOfRows
- UnionOfColumns
- DiagMat
- KroneckerMat
- DualKroneckerMat
- MulMat
- AddMat
- SubMat
- Compose
- NumberRows
- NumberColumns
- IsZeroMatrix
- IsDiagonalMatrix
- ZeroRows
- ZeroColumns

References

[hpa10] The homalg project authors. *The homalg project*, 2003-2010. http://homalg.math.rwth-aachen.de/. 3

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