Bessel Functions in other CRAN Packages

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Abstract

Why do I write yet another R package, when R itself has Bessel functions and several CRAN packages also have versions of these?

Short answer: I myself added the Bessel functions to R version 0.63, second half of 1998, but they have been seen to be limited for “large” x and / or large order ν.

Keywords: Bessel Functions, Accuracy, R.

1. Introduction

R itself has had the function `besselI()`,`besselJ()`,`besselK()` and `besselY()`, from very early on. Specifically, I myself added them to R version 0.63, in 1998. This helped quite a bit to attract people from computational finance to R in these early times. For some reason I must have been under the impression that the Fortran code I ported to C and interfaced with R to be state of the art at the time, even though I now doubt it.

However, they had shown deficiencies: First, they did only work for real (`double`) but not for *complex* arguments, even though the Bessel functions are well-defined on the whole complex plain. Second, for $x \approx 1500$ and larger, `besselI(x,nu, expon.scaled=TRUE)` jumped to zero, as I found, because of an overflow in the backward recursion (via difference equation), which I found elegantly to resolve (by re-scaling), for R2.9.0. However, the algorithm complexity is proportional to $\lfloor x \rfloor$, and for large x, a better algorithm has been desired for years. Hence, I had started experimenting with the two asymptotic expansions from Abramowitz and Stegun (1972).

The following R packages on CRAN (as of Jan.29, 2009) also provide Bessel functions:

gsl See Section refsec:gsl below

Rmpfr provides arbitrary precision Bessel functions of *integer* order $\nu =: n$ of the first kind only, $J_n(x) =$`jn(n,x)` and $Y_n(x) =$`yn(n,x)` (and `j0()`, `j1`, `y0`, `y1`) and—since MPFR version 3.0.0— the Airy function $Ai(x) =$`Ai(x)`.

```
> suppressPackageStartupMessages(require("Rmpfr"))
```

QRMlib Uses many ‘GSL’ (GNU Scientific Library) C functions in its own code, or, rather, has copy-pasted “Bessel-related” parts of GSL into its own ‘src/’ directory.

Notably ‘QRMlib/src/bessel.c’ is a copy (slightly modified to work as “standalone” in the QRMlib sources) of ‘GSL’’s ‘specfunc/bessel.c’ but has not been adapted to the latest

GSL sources. Further note that **QRMlib** only provides function `besselM3()()`: "M3" for the **m**odified Bessel function of the **3**rd kind, i.e., $K()$; note that it already has optional argument `logvalue=FALSE` and will call 'GSL''s `gsl_sf_bessel_lnKnu_e()` for `logvalue=TRUE`. Note that it calls different GSL routines for *integer* ν (=: n in that case) than for non-integer which presumably has at least computational advantages.

GeneralizeHyperbolic (todo)

ghyp (todo)

CircularDDM provides (a **Rcpp** and **gsl** based) function `besselzero(nu, k, kind)` to compute the first k zeros of the $J_\nu()$ (`kind=1`) and $Y_\nu()$ (`kind=0`) functions but fails to work for $I_\nu()$ (`kind=0`) where there is one zero for negative $\nu \in [-2k, -2k+1]$, $k = 1, 2, \ldots$.

2. Package 'gsl'

The R package **gsl** by Robin Hankin provides an R interface on a function-by-function basis to much of the GSL, the GNU Scientific Library. You get a first overview with

```
> library(gsl)

> ?bessel_Knu
> ?Airy
```

where the `?bessel_Knu` lists all "Bessel" functions and `?Airy` additionally the "Airy" functions $Ai()$ and $Bi()$ and their derivatives which are strongly related to the Bessel functions (and can be defined via them).

Indeed, the GSL and hence the R package **gsl** does contain quite an array of Bessel functions and the Airy functions, we can also get via

```
> igsl <- match("package:gsl", search())
> aB <- apropos("Bessel", where=TRUE); unname(aB)[names(aB) == igsl]

 [1] "bessel_I0"                "bessel_I0_scaled"
 [3] "bessel_I1"                "bessel_I1_scaled"
 [5] "bessel_In"                "bessel_In_array"
 [7] "bessel_In_scaled"         "bessel_In_scaled_array"
 [9] "bessel_Inu"               "bessel_Inu_scaled"
[11] "bessel_J0"                "bessel_J1"
[13] "bessel_Jn"                "bessel_Jn_array"
[15] "bessel_Jnu"               "bessel_K0"
[17] "bessel_K0_scaled"         "bessel_K1"
[19] "bessel_K1_scaled"         "bessel_Kn"
[21] "bessel_Kn_array"          "bessel_Kn_scaled"
[23] "bessel_Kn_scaled_array" "bessel_Knu"
```

```
[25] "bessel_Knu_scaled"      "bessel_Y0"
[27] "bessel_Y1"              "bessel_Yn"
[29] "bessel_Yn_array"        "bessel_Ynu"
[31] "bessel_i0_scaled"       "bessel_i1_scaled"
[33] "bessel_i2_scaled"       "bessel_il_scaled"
[35] "bessel_il_scaled_array" "bessel_j0"
[37] "bessel_j1"              "bessel_j2"
[39] "bessel_jl"              "bessel_jl_array"
[41] "bessel_jl_steed_array"  "bessel_k0_scaled"
[43] "bessel_k1_scaled"       "bessel_k2_scaled"
[45] "bessel_kl_scaled"       "bessel_kl_scaled_array"
[47] "bessel_lnKnu"           "bessel_sequence_Jnu"
[49] "bessel_y0"              "bessel_y1"
[51] "bessel_y2"              "bessel_yl"
[53] "bessel_yl_array"        "bessel_zero_J0"
[55] "bessel_zero_J1"         "bessel_zero_Jnu"
```

```
> aA <- apropos("Airy",   where=TRUE); unname(aA)[names(aA) == igsl]
```

```
 [1] "airy_Ai"               "airy_Ai_deriv"         "airy_Ai_deriv_scaled"
 [4] "airy_Ai_scaled"        "airy_Bi"               "airy_Bi_deriv"
 [7] "airy_Bi_deriv_scaled" "airy_Bi_scaled"        "airy_zero_Ai"
[10] "airy_zero_Ai_deriv"   "airy_zero_Bi"          "airy_zero_Bi_deriv"
```

Features (and drawbacks):

- only real 'x', not complex
- provides separate functions for *integer* and *fractional* ν where the latter should be more general than the former (untested in detail though).
- For *fractional* ν, the relevant, i.e., interesting functions are

  ```
  bessel_Jnu       (nu, x, give=FALSE, strict=TRUE)

  bessel_Ynu       (nu, x, give=FALSE, strict=TRUE)

  bessel_Inu       (nu, x, give=FALSE, strict=TRUE)
  bessel_Inu_scaled(nu, x, give=FALSE, strict=TRUE)

  bessel_Knu       (nu, x, give=FALSE, strict=TRUE)
  bessel_Knu_scaled(nu, x, give=FALSE, strict=TRUE)
  bessel_lnKnu     (nu, x, give=FALSE, strict=TRUE)
  ```

  where the `*_scaled()` version of each corresponds to our functions `expon.scaled=TRUE`.
- For fractional nu , the (only) interesting functions are

```
> lst <- ls(patt="bessel_.*nu", pos="package:gsl")
> l2 <- sapply(lst, function(.) args(get(.)), simplify=FALSE)
> lnms <- setNames(format(lst), lst)
> arglst <- lapply(lst, ## a bit ugly, using deparse(.)
+     function(nm) sub(" *$","", sub("^function", lnms[[nm]], deparse(l2[[nm]])[[1]])
> .tmp <- lapply(arglst, function(.) cat(format(.),"\n"))

bessel_Inu          (nu, x, give = FALSE, strict = TRUE)
bessel_Inu_scaled   (nu, x, give = FALSE, strict = TRUE)
bessel_Jnu          (nu, x, give = FALSE, strict = TRUE)
bessel_Knu          (nu, x, give = FALSE, strict = TRUE)
bessel_Knu_scaled   (nu, x, give = FALSE, strict = TRUE)
bessel_Ynu          (nu, x, give = FALSE, strict = TRUE)
bessel_lnKnu        (nu, x, give = FALSE, strict = TRUE)
bessel_sequence_Jnu (nu, v, mode = 0, give = FALSE, strict = TRUE)
bessel_zero_Jnu     (nu, s, give = FALSE, strict = TRUE)
```

where the `*_scaled()` version of each function corresponds to our functions with option `expon.scaled=TRUE`.

- `bessel_Inu_scaled()` works for large x, comparably to our `BesselI(.)` which give warnings about accuracy loss here :

```
>    x <- (1:500)*50000; b2 <- BesselI(x, pi, expo=TRUE)
>    b1 <- bessel_Inu_scaled(pi, x)
>    all.equal(b1,b2,tol=0) ## "Mean relative difference: 1.544395e-12"

[1] "Mean relative difference: 2.226772e-12"

>    ## the accuracy is *as* limited (probably):
>    b1 <- bessel_Inu_scaled(pi, x, give=TRUE)
>    summary(b1$err)

     Min.   1st Qu.    Median      Mean   3rd Qu.      Max.
8.299e-08 9.580e-08 1.173e-07 1.606e-07 1.655e-07 1.856e-06
```

  where the GSL (info) manual says that `err` is an *absolute* error estimate, hence for *relative* error estimates, we look at

```
>     range(b1$err/ b1$val)

[1] 0.001040159 0.001040161
```

  So, we see that either the error estimate is too conservative, or the results only have 3 digit accuracy.

- $J_\nu(.)$: Here (also), the GSL employs different algorithms in different regions, notably also several asymptotic formula. When $x < \nu$, notably $0 \approx x \ll \nu$, it does not seem to be ok, in the the "left tail", returning `NaN`, for moderate ν:

```
> bessel_Jnu(100,  2^seq(-5,1, by=1/4))

 [1]           NaN           NaN           NaN           NaN           NaN
 [6] 1.098354e-301 3.685445e-294 1.236620e-286 4.149362e-279 1.392272e-271
[11] 4.671585e-264 1.567474e-256 5.259330e-249 1.764625e-241 5.920564e-234
[16] 1.986357e-226 6.663900e-219 2.235461e-211 7.498243e-204 2.514703e-196
[21] 8.431829e-189 2.826353e-181 9.469925e-174 3.171070e-166 1.060953e-158

> bessel_Jnu( 20,  2^seq(-50,-40, by=1/2))

 [1]           NaN           NaN           NaN           NaN           NaN
 [6]           NaN 4.217737e-308 4.318963e-305 4.422618e-302 4.528761e-299
[11] 4.637451e-296 4.748750e-293 4.862720e-290 4.979426e-287 5.098932e-284
[16] 5.221306e-281 5.346617e-278 5.474936e-275 5.606335e-272 5.740887e-269
[21] 5.878668e-266

> bessel_Jnu(  5,  2^seq(-210,-200, by=.5))

 [1]           NaN           NaN           NaN           NaN           NaN
 [6]           NaN           NaN           NaN           NaN           NaN
[11]           NaN           NaN           NaN           NaN           NaN
[16]           NaN 2.373412e-308 1.342605e-307 7.594919e-307 4.296335e-306
[21] 2.430374e-305
```

giving `NaN` instead of just underflowing to zero. However, looking at the phenomenon shows that it is only because of the **gsl**'s default optional argument `strict = TRUE`: The underflow to zero which no longer allows the error to be controlled (and returned in `err` when `give = TRUE`), giving `status = 15` here:

```
> as.data.frame(bessel_Jnu( 20,  2^seq(-50,-40, by=1/2), give=TRUE, strict=FALSE))

             val           err status
1   0.000000e+00 2.225074e-308     15
2   0.000000e+00 2.225074e-308     15
3   0.000000e+00 2.225074e-308     15
4   0.000000e+00 2.225074e-308     15
5   0.000000e+00 2.225074e-308     15
6   0.000000e+00 2.225074e-308     15
7  4.217737e-308 2.371515e-322      0
8  4.318963e-305 2.397503e-319      0
9  4.422618e-302 2.455046e-316      0
10 4.528761e-299 2.513967e-313      0
11 4.637451e-296 2.574303e-310      0
12 4.748750e-293 2.636086e-307      0
13 4.862720e-290 2.699352e-304      0
14 4.979426e-287 2.764136e-301      0
15 5.098932e-284 2.830476e-298      0
16 5.221306e-281 2.898407e-295      0
```

```
17 5.346617e-278 2.967969e-292      0
18 5.474936e-275 3.039200e-289      0
19 5.606335e-272 3.112141e-286      0
20 5.740887e-269 3.186832e-283      0
21 5.878668e-266 3.263316e-280      0
```

If we do use `strict = FALSE`, consequently, all is fine:

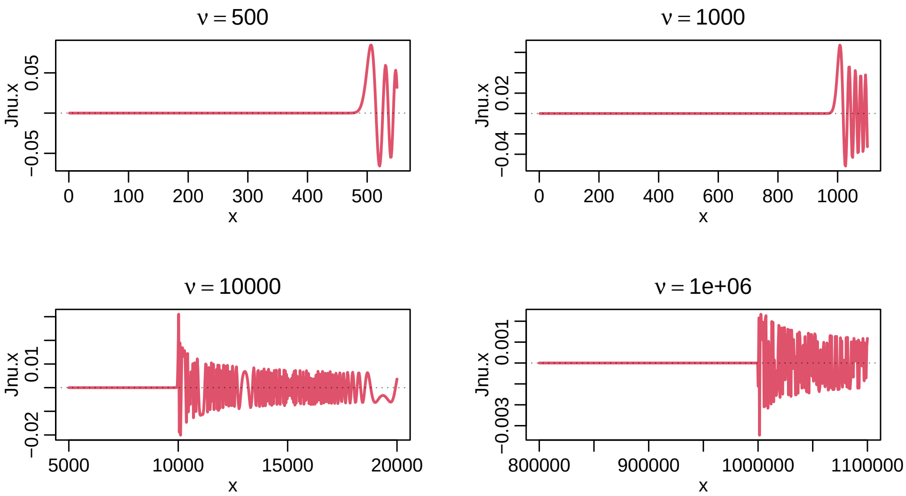
```
> gslJ <- function(nu, f1 = .90, f2 = 1.10, nout = 512, give=FALSE, strict=FALSE) {
+     stopifnot(is.numeric(nu), length(nu) == 1, nout >= 1, f1 <= 1, f2 >= 1)
+     x <- seq(f1*nu, f2*nu, length.out = nout)
+     list(x=x, Jnu.x = bessel_Jnu(nu, x, give=give, strict=strict))
+ }
> plJ <- function(nu, f1 =.90, f2=1.10, nout=512,
+                 col=2, lwd=2, main = bquote(nu == .(nu)), ...) {
+     dJ <- gslJ(nu, f1=f1, f2=f2, nout=nout)
+     plot(Jnu.x ~ x, data=dJ, type="l", col=col, lwd=lwd, main=main, ...)
+     abline(h=0, lty=3, col=adjustcolor(1, 0.5))
+     invisible(dJ)
+ }
> sfsmisc::mult.fig(4)
> plJ(500, f1=0)
> r1k <- plJ(1000, f1=0)
> head(as.data.frame(r1k)) # all 0 now (NaN's for  'strict=TRUE' !!)
```

```
          x Jnu.x
1  0.000000   NaN
2  2.152642     0
3  4.305284     0
4  6.457926     0
5  8.610568     0
6 10.763209     0
```

```
> r10k <- plJ(10000, f1=0.5, f2=2)
> str( with(r10k, x[!is.finite(Jnu.x)]) ) # empty; had all NaN upto x = 8317
```

```
 num(0)
```

```
> r1M <- plJ(1e6, f1=0.8)
```



3. Session Info

```
> toLatex(sessionInfo(), locale=FALSE)
```

- R version 4.4.1 (2024-06-14), `x86_64-pc-linux-gnu`
- Running under: `Fedora Linux 40 (Forty)`
- Matrix products: default
- BLAS: `/r/app/R/R-4.4.1-isg/lib64/R/lib/libRblas.so`
- LAPACK: `/usr/lib64/liblapack.so.3.12.0`
- Base packages: base, datasets, grDevices, graphics, methods, stats, utils
- Other packages: Bessel 0.6-1, Rmpfr 0.9-5, gmp 0.7-4, gsl 2.1-8
- Loaded via a namespace (and not attached): compiler 4.4.1, sfsmisc 1.1-18, tools 4.4.1

Date (run in R): 2024-07-29

References

Abramowitz M, Stegun IA (1972). *Handbook of Mathematical Functions*. Dover Publications, N. Y. URL https://en.wikipedia.org/wiki/Abramowitz_and_Stegun.

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